

Construction of a Free Lévy Process as high-dimensional limit of a Brownian Motion on the Unitary Group

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Abstract

It is well known that freeness appears in the high-dimensional limit of independence for matrices. Thus, for instance, the additive free Brownian motion can be seen as the limit of the Brownian motion on hermitian matrices. More generally, it is quite natural to try to build free Lévy processes as high-dimensional limits of classical matricial Lévy processes.

We will focus here on one specific such construction, discussing and generalizing the work done previously by Biane in [1], who has shown that the (classical) Brownian motion on the Unitary group $U(d)$ converges to the free multiplicative Brownian motion when d goes to infinity. We shall first recall that result and give an alternative proof for it. We shall then see how this proof can be adapted in a more general context in order to get a free Lévy process on the dual group (in the sense of Voiculescu) $U\langle n \rangle$. This result will actually amount to a truly noncommutative limit theorem for classical random variables, of the which Biane's result constitutes the case $n = 1$.

1 Biane's result about the Brownian motion on the Unitary group

In all the following, we assume that a unital noncommutative probability space (A, ϕ) be given. Let us remind what we mean by that definition: a unital noncommutative probability space is a couple (A, ϕ) where A is a unital $*$ -algebra and ϕ is a linear functional on A such that $\phi(a^*a) \geq 0$ for each $a \in A$ and $\phi(1) = 1$.

We will also write by δ_{ab} Kronecker's symbol, which is equal to 0 when $a \neq b$ and is equal to 1 when $a = b$. Let us recall following definitions and result:

Definition 1. We denote by $(\nu_t)_{t \geq 0}$ the same family of measures on the unit circle as in [1], ie ν_t is the only probability measure such that $\xi_{\nu_t}(z) = z \exp[\frac{1}{2} \frac{1+z}{1-z}]$, where ξ_{ν_t} is the inverse function of $\frac{\psi_{\nu_t}}{1+\psi_{\nu_t}}$ and $\psi_{\nu_t} = \int \frac{z\zeta}{1-z\zeta} d\nu_t(\zeta)$ where the integration is done on the unit circle.

Definition 2. A free multiplicative Brownian motion is a family $(U_t)_{t \geq 0}$ such that:

- For every $0 \leq t_1 < t_2 < \dots < t_n$, the family $(U_{t_1}, U_{t_2}U_{t_1}^{-1}, \dots, U_{t_n}U_{t_{n-1}}^{-1})$ is free.

- For every $0 \leq s < t$ the element $U_t U_s^{-1}$ has a distribution ν_{t-s} .

In his paper [1], Biane proved that Brownian motion on the group $U(d)$ converges, as d goes to infinity, towards a multiplicative free Brownian motion. To do this, he proves first the convergence of the marginals using representation theory arguments and secondly the freeness of the increments. We suggest here that there is an other way to prove the convergence of the marginals based on the Itô formula.

Let us first observe that the Brownian motion on the Unitary group $U(d)$ can be defined as the unique solution of:

$$dU_t^{(d)} = i d H_t U_t^{(d)} - \frac{1}{2} U_t^{(d)} dt$$

with initial condition $U_0 = I$. Note that we denote by i the complex number, so as to differentiate it from the index i . In the same way we write d the differential operator so as to distinguish it from the size of the matrices. In this equation, we have noted by H_t a Brownian motion on hermitian matrices defined by:

- The family $(H_{ij}(t))_{1 \leq i \leq j \leq d}$ is an independent family of random variables
- For $1 \leq i \leq d$, we have $H_{ii}(t)$ a gaussian variable $\mathcal{N}(0, \frac{1}{d})$
- For $1 \leq k \leq j \leq d$, we have $H_{kj}(t) = H_{kj}^{(1)}(t) + i H_{kj}^{(2)}(t)$ with $H_{kj}^{(1)}(t)$ and $H_{kj}^{(2)}(t)$ two independent gaussian variables $\mathcal{N}(0, \frac{1}{2d})$
- The matrix $H(t)$ is hermitian for each t .

In particular this means that each entry of H_t is of variance $1/d$.

Note: we shall omit the exponent (d) when there is no confusion possible. Let us now denote by f_{k_1, \dots, k_r} the following function of t :

$$f_{k_1, \dots, k_r} = \mathbb{E} \left[tr \left(U_t^{k_1} \right) \dots tr \left(U_t^{k_r} \right) \right]$$

where the trace is normalized¹ by $1/d$. We will find a differential equation involving those functions.

Lemma 1. *We have the following formula:*

$$\begin{aligned} d(U_{i_1 j_1} \dots U_{i_r j_r}) &= \text{martingale} - \frac{1}{2} \sum_{k=1}^r U_{i_1 j_1} \dots U_{i_r j_r} dt \\ &\quad - \frac{dt}{d} \sum_{1 \leq p < q \leq r} U_{i_1 j_1} \dots U_{i_p j_q} \dots U_{i_q j_p} \dots U_{i_r j_r} \end{aligned}$$

¹The convention we adopt in this paper is following: whenever we mean the normalized trace, we write tr and we write Tr whenever we speak of the usual trace.

This means that the non-martingale part is constituted by two terms, the first one where nothing is changed in the indices and the second one where you have switched two indices: j_q replaces j_p and j_p replaces a j_q .

Proof. This is obtained by using Itô's formula and by reasoning for each element in the matrix, because:

$$\mathbf{d}(U_{i_1 j_1} \dots U_{i_r j_r}) = \sum_{k=1}^r U_{i_1 j_1} \dots (\mathbf{d}U_{i_k j_k}) \dots U_{i_r j_r} + \sum_{1 \leq k < l \leq r} \prod_{s \neq k, l} U_{i_s j_s} \mathbf{d}[U_{i_k j_k}, U_{i_l j_l}]$$

The $[\cdot, \cdot]$ denotes the quadratic variation. We remark that:

$$\forall i, j, \mathbf{d}U_{ij}(t) = \mathbf{i} \sum_{r=1}^d \mathbf{d}H_{ir} U_{rj} - \frac{1}{2} U_{ij} \mathbf{d}t$$

and

$$\mathbf{d}[H_{i_k r_k}, H_{i_l r_l}] = \mathbf{d}[H_{i_k r_k}^{(1)} + \mathbf{i} H_{i_k r_k}^{(2)}, H_{i_l r_l}^{(1)} + \mathbf{i} H_{i_l r_l}^{(2)}]$$

But we know that the quadratic variation of two processes is zero if they are independent. Thus, $\mathbf{d}[H_{i_k r_k}, H_{i_l r_l}]$ is equal to:

- If $i_k = i_l$ and $j_l = j_k$, $\mathbf{d}[H_{i_k j_k}, H_{i_l j_l}] = \frac{1}{2d} - \frac{1}{2d} = 0$
- If $i_k = j_l$ and $j_k = i_l$, $\mathbf{d}[H_{i_k j_k}, H_{i_l j_l}] = \frac{1}{2d} + \frac{1}{2d} = \frac{1}{d}$
- And it is equal to zero in all other cases.

And thus, the quadratic variation can be expressed as:

$$\begin{aligned} \mathbf{d}[U_{i_k j_k}, U_{i_l j_l}] &= \mathbf{i} \sum_{r_l, r_k=1}^d U_{r_k j_k} U_{r_l j_l} \mathbf{d}[H_{i_k r_k}, H_{i_l r_l}] + \text{martingale} \\ &= \mathbf{i} U_{i_l j_k} U_{i_k j_l} \end{aligned}$$

□

When we take the expectation, the martingale part vanishes. If we expand f_{k_1, \dots, k_r} , we get:

$$f_{k_1, \dots, k_r} = \frac{1}{d^r} \mathbb{E} \left[\sum_{\substack{i_1^1, \dots, i_{k_1}^1 = 1 \\ i_1^r, \dots, i_{k_r}^r}}^d U_{i_1^1 i_2^1} \dots U_{i_{k_1}^1 i_1^1} \dots U_{i_1^r i_2^r} \dots U_{i_{k_r}^r i_1^r} \right]$$

To get a system of differential equations we will use the former formula that we have obtained thanks to Itô's Lemma. Especially we must see how the

last term, switching p and q , can be rewritten in terms of the functions f_{k_1, \dots, k_r} . There are actually two cases to study: first when p and q come from the same trace and second when they come from different traces.

When they come from the same trace: If for instance p and q both come from the m^{th} trace, the contribution of this trace is of the kind:

$$\frac{1}{dr} \dots U_{i_1^m i_2^m} \dots U_{i_p^m i_{p+1}^m} \dots U_{i_q^m i_{q+1}^m} \dots U_{i_m^m i_1^m} \dots$$

So when we do the switching it yields:

$$\frac{1}{dr} \dots U_{i_1^m i_2^m} \dots U_{i_p^m i_{q+1}^m} \dots U_{i_q^m i_{p+1}^m} U_{i_{q+1}^m i_{q+2}^m} \dots U_{i_m^m i_1^m}$$

And when we sum over all those indices we see that we actually get: $df_{k_1, \dots, k_m - (q-p), q-p, \dots, k_r}$, ie the switching has produced one more trace.

When they come from two different traces: We shall here suppose that p comes from the u^{th} trace and q comes from the v^{th} trace, with $u < v$. The contribution of those two traces are:

$$\frac{1}{dr} \dots U_{i_1^u i_2^u} \dots U_{i_p^u i_{p+1}^u} \dots U_{i_{k_u}^u i_1^u} \dots U_{i_1^v i_2^v} \dots U_{i_q^v i_{q+1}^v} \dots U_{i_{k_v}^v i_1^v} \dots$$

Switching p and q yields to:

$$\frac{1}{dr} \dots U_{i_1^u i_2^u} \dots U_{i_p^u i_{q+1}^v} \dots U_{i_{k_u}^u i_1^u} \dots U_{i_1^v i_2^v} \dots U_{i_q^v i_{p+1}^u} U_{i_{q+1}^v i_{q+2}^v} \dots U_{i_{k_v}^v i_1^v} \dots$$

And so if we sum over all indices we see that we get $\frac{1}{d} f_{k_1, \dots, k_u + k_v, \dots, k_r}$, ie we have merged two traces together.

So, if we put it all together we see by using Lemma 1 that the system of differential equations we get is:

$$\begin{aligned} f'_{k_1, \dots, k_r} &= -\frac{k_1 + \dots + k_r}{2} f_{k_1, \dots, k_r} - \sum_{\kappa=1}^r \sum_{l=1}^{k_\kappa} (k_\kappa - l) f_{k_1, \dots, k_\kappa - l, l, \dots, k_r} \\ &- \frac{1}{d^2} \sum_{1 \leq \kappa < \lambda \leq r} \sum_{p=1}^{k_\kappa} \sum_{q=1}^{k_\lambda} f_{k_1, \dots, k_\kappa + k_\lambda, \dots} \end{aligned}$$

Let us observe here that we have a nice combinatorial structure for these equations. Indeed, we can interpret (k_1, \dots, k_r) as an integer partition for the integer $k_1 + \dots + k_r$. By doing so, we see that the equation only involves partitions for the same integer because we either split an integer into two parts or we merge two integers into one. These equations thus have the same structure as the equations in Proposition 2.3 in [5] via the identification between a permutation and the length of the cycles of its canonical decomposition.

Let us also note that an integer l has only finitely many partitions.² So that means that each function is involved in a system of finitely many linear differential equations with fixed initial conditions.

What can we say about the convergence of this family of functions? We actually have that for each $r \geq 1$ and every $k_1, \dots, k_r \geq 1$, the function $f_{k_1, \dots, k_r}^{(d)}$ converges, as d goes to infinity, towards a function f_{k_1, \dots, k_r} verifying:

$$f'_{k_1, \dots, k_r} = -\frac{k_1 + \dots + k_r}{2} f_{k_1, \dots, k_r} - \sum_{\kappa=1}^r \sum_{l=1}^{k_\kappa} (k_\kappa - l) f_{k_1, \dots, k_\kappa - l, l, \dots, k_r}$$

Indeed, let us fix such a partition $k_1 + \dots + k_r = k$. If we note $P(k) := \{(k_1, \dots, k_r) | r \geq 0, k_1 + \dots + k_r = k\}$ the set of partitions of the integer k , we have just shown that this set is finite. The function $f_{k_1, \dots, k_r}^{(d)}$ thus only shows up in a finite number of linear differential equations with constant coefficients. This finite number of differential equations can be rewritten in a matricial form: let $\Phi_t^{(d)}$ be a vector in $\mathbb{C}^{\#P(k)}$ consisting of all functions $f_{p_1, \dots, p_l}^{(d)}$ where p_1, \dots, p_l is a partition of the same integer k . Then $\Phi^{(d)}$ is solution of a differential equation of the form:

$$(\Phi^{(d)})' = A^{(d)} \Phi^{(d)}$$

where $A^{(d)}$ is a (constant) matrix formed with the coefficients of our differential equations. It is well-known that $\Phi^{(d)}$ is thus of the form $\Phi^{(d)} = \Phi_0^{(d)} e^{A^{(d)} t}$. But the coefficients of the equations for $f^{(d)}$, namely $A^{(d)}$, converge towards the coefficients for the equation of f , namely A , and thus $\Phi^{(d)}$ converges towards Φ , or in other words, $f_{k_1, \dots, k_r}^{(d)}$ converges towards f_{k_1, \dots, k_r} .

We will now denote by F_{k_1, \dots, k_r} the function $\phi(u_t^{k_1}) \dots \phi(u_t^{k_r})$ where u is here a free multiplicative Brownian motion. To prove the convergence of the marginals it will be enough to prove that the family of functions F verifies the differential equations system:

$$F'_{k_1, \dots, k_r} = -\frac{k_1 + \dots + k_r}{2} F_{k_1, \dots, k_r} - \sum_{\kappa=1}^r \sum_{l=1}^{k_\kappa} (k_\kappa - l) F_{k_1, \dots, k_\kappa - l, l, \dots, k_r}$$

Indeed, if we have proven it, then it implies that for all $r \geq 1$ and all $0 \leq t_1 \leq \dots \leq t_r$ the function $f_{t_1, \dots, t_r}^{(d)}$ converges towards $F_{t_1 \dots t_r}$ when d

²Without going into the details of the theory of integer partitions, we may find a gross upper bound for this number in the following way: A partition of l cannot have more than l parts. So let's consider a line consisting of $l + l - 1 = 2l - 1$ boxes. We then put crosses in $l - 1$ boxes. Each such cross helps separate two parts of the partition. For instance:



represents the partition $(1, 1, 2)$ of the integer 4. Hence we see that the number of such partitions is bounded by $\binom{2l-1}{l-1}$, which is finite.

goes to infinity. In particular, if we take $r = 1$, we see that we have the convergence of the marginals (in moments).

In order to prove that formula we must remark that a free multiplicative Brownian motion is given by a free stochastic equation with initial conditions $u_0 = 1$ (1 is the unit element of A):

$$\mathbf{d}u_t = \mathbf{i} \mathbf{d}X_t u_t - \frac{1}{2} u_t \mathbf{d}t$$

where X_t is a free additive Brownian motion. This result is stated in [1]'s Theorem 2. We will simplify the calculations by putting $V_t := e^{t/2} u_t$. Using the free analogue of Itô's Lemma (see e.g. [4], Theorem 5), Biane demonstrated following formula

$$\mathbf{d}V_t^n = \mathbf{i} \sum_{k=0}^n V_t^k \mathbf{d}X_t V_t^{n-k} - \sum_{k=1}^{n-1} k V_t^k \phi(V_t^{n-k}) \mathbf{d}t$$

In other words this means:

$$\mathbf{d}u_t^n = \mathbf{i} \sum_{k=0}^n u_t^k \mathbf{d}X_t u_t^{n-k} - \sum_{k=1}^{n-1} k u_t^k \phi(u_t^{n-k}) \mathbf{d}t - \frac{n}{2} u_t^n \mathbf{d}t$$

Taking the trace of it we obtain:

$$\phi(u_t^n)' = - \sum_{k=1}^{n-1} k \phi(u_t^k) \phi(u_t^{n-k}) - \frac{n}{2} \phi(u_t^n)$$

And so it finally yields the following system of differential equations:

$$F'_{k_1, \dots, k_r} = - \frac{k_1 + \dots + k_r}{2} F_{k_1 \dots k_r} - \sum_{\kappa=1}^r \sum_{p=1}^{k_\kappa-1} p F_{k_1, \dots, p, k_\kappa-p, \dots, k_r}$$

And this is exactly the system we wanted because $F_{k_1, \dots, p, k_\kappa-p, \dots} = F_{k_1, \dots, k_\kappa-p, p, \dots}$. To put it in a nutshell: we were able to reprove Biane's result by using a different method (by comparing systems of differential equations) to prove the convergence of marginals. The freeness of the increments can still be proven as did Biane but it will also follow from the results of section 4. We will now try to use that alternative method to generalize Biane's result. To do that we will need the concept of dual groups.

2 Dual groups in the sense of Voiculescu and Lévy processes

We will here briefly introduce dual groups as they were first defined by Voiculescu. For more information on this subject one can read [10]. In the sequel we denote by \sqcup the free product of unital $*$ -algebras.

Definition 3 (Dual semigroups). A (unital) dual semigroup is a triple (B, Δ, δ) where B is a $*$ -algebra and $\Delta : B \rightarrow B \sqcup B$ and $\delta : B \rightarrow \mathbb{C}$ are $*$ -homomorphisms such that

$$(\Delta \sqcup id_B) \circ \Delta = (Id_B \sqcup \Delta)$$

$$(\delta \sqcup id_B) \circ \Delta = id_B = (Id_B \sqcup \delta) \circ \Delta$$

The former property is called coassociativity, whereas the latter is the counit property.

When considering the free product $B \sqcup B$, in order to differentiate between elements coming from the B on the left and elements coming from the B on the right, we will talk about the left and the right legs of $B \sqcup B$.

We shall be in this paper particularly interested in one dual group:

Definition 4 (Unitary Dual Group). For $n \geq 1$, we call Unitary Dual Group the dual group $(U\langle n \rangle, \Delta, \delta)$ defined by:

- The $*$ -algebra $U\langle n \rangle$ is generated by n^2 generators $(u_{ij})_{1 \leq i, j \leq n}$ verifying the relations:

$$\forall 1 \leq i, j \leq n \quad \sum_{k=1}^n u_{ki}^* u_{kj} = \delta_{ij} = \sum_{k=1}^n u_{ik} u_{jk}^*$$

- The coproduct is given by:

$$\Delta u_{ij} = \sum_k u_{ik}^{(1)} u_{kj}^{(2)}$$

where the exponent (1) (resp. (2)) indicates that the element is taken from the left (resp. right) leg of $U\langle n \rangle \sqcup U\langle n \rangle$.

- The counit is given by: $\delta u_{ij} = \delta_{ij}$.

Dual semigroups are particularly useful to define free Lévy processes in the most general case.

Definition 5 (Lévy processes). We shall assume that we have a dual semigroup (B, Δ, δ) and some unital noncommutative probability space (A, ϕ) . A free (resp. tensor independent) Lévy process on the semigroup B over the noncommutative probability space (A, ϕ) is a family $(j_{s,t})_{0 \leq s \leq t}$ of $*$ -homomorphisms from B to A such that:

- (Increment Property) For every $0 \leq s \leq t \leq r$ we have:

$$(j_{st} \sqcup j_{tr}) \circ \Delta = j_{sr}$$

- (Stationarity) We have for every $0 \leq s \leq t$: $j_{0,t-s} = j_{s,t}$
- (Freeness of the Increments) For every $0 \leq t_1 \leq t_2 \leq \dots \leq t_{n+1}$, the increments $j_{t_1 t_2}, \dots, j_{t_n t_{n+1}}$ are free (resp. tensor independent).
- (Weak continuity) For each $b \in B$ and each $s \geq 0$, we have: $\lim_{t \rightarrow s^+} \phi \circ j_{s,t}(b) = \delta(b)$

How can these concepts be applied in our case? We could generalize Biane's question by taking $U_t^{(d)}$ a Brownian motion on the Unitary Group $U(nd)$, where n is a fixed integer. The matrix $U_t^{(d)}$ can be decomposed in n^2 blocks of size $d \times d$. In the sequel of the article we will denote by $[U_t^{(d)}]_{ij}$ the $(i, j)^{\text{th}}$ block of our Brownian motion. For each d we thus get a quantum stochastic process on the Dual Unitary Group by setting for $0 \leq s \leq t$:

$$\begin{aligned} j_{st}^{(d)} &: U\langle n \rangle \rightarrow (A, \phi) \\ u_{ij} &\mapsto [U_t^{(d)}]_{ij} \end{aligned}$$

We will in the sequel of the article omit the exponent $\langle d \rangle$ whenever no confusion can arise.

The question that is natural to ask and that generalizes Biane's result is whether or not j_{st} converges to a Lévy process on $U\langle n \rangle$ in the limit when d goes to infinity.

We will show that we have following result

Theorem 1 (Main Theorem). *We assume that ϕ is tracial.*

Let $X = (X_{ij})_{1 \leq i, j \leq n}$ be a matrix whose entries are free stochastic variables verifying that:

- *For each i , X_{ii} is an additive free Brownian motion.*
- *For every $i \neq j$, $X_{ij} = X_{ij}^{(1)} + iX_{ij}^{(2)}$ with $\sqrt{2}X_{ij}^{(1)}$ and $\sqrt{2}X_{ij}^{(2)}$ who are two additive free Brownian motions who are free one with another.*
- *For each i, j we have $X_{ij} = X_{ji}^*$.*
- *The family $(X_{ij})_{1 \leq i \leq j \leq n}$ is free.*

Let also $\Psi = (\Psi_{ij})$ be a free stochastic process defined by the free stochastic equation with initial condition $\Psi_0 = I$:

$$d\Psi_t = \frac{i}{\sqrt{n}} dX_t \Psi_t - \frac{1}{2} \Psi_t dt$$

Through Ψ we may define a free Lévy process J through³:

$$\begin{aligned} J_{st} &: U\langle n \rangle \rightarrow (A, \phi) \\ u_{ij} &\mapsto \Psi_{ij} \end{aligned}$$

Then, $(j_{st}^{(d)})$ converges towards (J_{st}) as d goes to infinity.

3 Convergence of the marginals

We will first study the convergence of the marginals. Hence we will fix in this section a $t \geq 0$. To prove such a convergence we must study the moments of the type $\phi \circ j_{0t}(u_{i_1 j_1}^{\epsilon_1} \dots u_{i_r j_r}^{\epsilon_r})$, where $\epsilon_1, \dots, \epsilon_r \in \{\emptyset, *\}$. For convenience, we will identify \emptyset with 0 and $*$ with 1. We will use exactly the same method as in the first section but, because there are n^2 variables, we will have many more indices.

3.1 Notations

We consider the dual group $U\langle n \rangle$ which is generated by n^2 variables. We will need to introduce some notations to describe all the indices that will be involved.

From now on and until the end of the paper, when we have a matrix $M \in \mathcal{M}_{nd}(\mathbb{C})$, we will denote:

- by M_{ij} the (i, j) -matrix entry of M .
- by $[M]_{ij}$ the (i, j) -block of size $d \times d$ of the matrix M

We denote by $[\mathcal{I}]$ the set $[\mathcal{I}] = \{1, \dots, n\}^2 \times \{0, 1\}$. For such a triple $\alpha = (i, j, \epsilon)$, we will denote $[U]_\alpha$ the $d \times d$ block $[U]_{ij}^\epsilon$, where we identify $\epsilon = 1$ with $*$ and $\epsilon = 0$ with \emptyset .

We denote by \mathcal{I} the set $\mathcal{I} = \{1, \dots, nd\}^2 \times \{0, 1\}$. For such a triple $\rho = (\mu, \nu, \omega)$, we will denote U_ρ the coefficient $U_{\mu\nu}$ if $\omega = 0$ and the coefficient $\bar{U}_{\mu\nu}$ if $\omega = 1$.

When Ψ is in $\mathcal{M}_n(\mathcal{A})$, with \mathcal{A} a $*$ -algebra, we denote by Ψ_α the element Ψ_{ij}^ϵ .

³By calculating $d(\sum_k \Psi_{ki}^* \Psi_{kj})$ we find zero. Moreover, when we calculate $d(\sum_k \Psi_{ik} \Psi_{jk}^*)$ we find a free stochastic differential equation that is verified by the constant δ_{ij} . By unicity of the solution (see e.g. [4][Theorem 4]), we have that $\sum_k \Psi_{ik} \Psi_{jk}^* = \delta_{ij}$. Thus J_{st} respects the defining relations of $U\langle n \rangle$

3.2 A system of differential equations for the Brownian motion on $U(nd)$

To achieve our purpose we need to consider the family of functions (as always, we will omit the exponents everytime we may do so without risk):

$$\begin{aligned} & \gamma_{\alpha_{11}, \dots, \alpha_{k_1 1}; \dots; \alpha_{1r}, \dots, \alpha_{k_r r}}^{(d)} \\ &= \mathbb{E}[tr([U]_{\alpha_{11}} \dots [U]_{\alpha_{k_1 1}}) \dots tr(\dots [U]_{\alpha_{k_r r}})] \end{aligned}$$

where $r \geq 1; k_1, \dots, k_r \in \mathbb{N}, \alpha_{kl} \in [\mathcal{I}]$.

In other words, we take functions very similar to what we had before in the simpler case of the convergence to Biane's result. They still are the product of traces⁴. The difficulty arises here from the fact that we consider blocks and that we thus have to consider all possible products of the blocks and their adjoints. The indices we use specify which U_{ij} appear and if they have a $*$ or not and the semicolumns separate two traces. We will, as previously, try to find a system of differential equations. Let us fix the indices $\alpha_{11} \dots \alpha_{k_r r}$. Again, we apply Lemma 1 in order to calculate the differential equation. For the sake of simplicity let us first observe what happens if we suppose that there are no $*$ in our function and we will later explain how to get the general case. As previously we treat separately the case where the switch occurs inside a same trace and the case where it affects two distinct traces. **The switch occurs in the same trace:** Let's say that the switch is between p and q inside the κ^{th} trace. Then, when we develop the traces, we see that the contribution of this trace, after the switch, is of the type:

$$\mathbb{E}\left[\sum_{s_{11} \dots s_{k_r r}} \dots U_{(i_{p\kappa}-1)d+s_{p\kappa}, (j_{q\kappa}-1)d+s_{q\kappa}} \dots U_{(i_{q\kappa}-1)d+s_{q\kappa}, (j_{p\kappa}-1)d+s_{p\kappa}} \dots\right]$$

As we could have expected the κ^{th} trace will be divided into two distinct traces: we get $d\gamma_{\dots; i_{1\kappa} j_{1\kappa}, \dots, i_{p\kappa} j_{q\kappa}, i_{q+1, \kappa} j_{q+1, \kappa}, \dots, i_{p+1, \kappa} j_{p+1, \kappa}, \dots, i_{q\kappa} j_{p\kappa}; \dots}$ (we recall that the normalization constant we now use for the trace is $1/d$).

The switch concerns two distinct traces: If we do the calculations, we see that we reunite these two traces and that we get a multiplicative factor $1/d$.

So, if we put it all together (in the case we have no $*$ at all), the equation we will have is:

$$\begin{aligned} & \gamma'_{\alpha_{11}, \dots, \alpha_{k_1 1}; \dots; \alpha_{k_r r}} \\ &= -\frac{k_1 + \dots + k_r}{2} \gamma_{\alpha_{11}, \dots, \alpha_{k_1 1}; \dots; \alpha_{k_r r}} \\ & - \sum_{\kappa=1}^r \sum_{1 \leq p < q \leq k_\kappa} \frac{1}{n} \gamma_{\dots; \alpha_{1\kappa}, \dots, (i_{p\kappa} j_{q\kappa} 0), \alpha_{q+1, \kappa}, \dots, \alpha_{p+1, \kappa}, \dots, (i_{q\kappa} j_{p\kappa} 0); \dots} \\ & + \mathcal{O}\left(\frac{1}{d^2}\right) \end{aligned}$$

⁴The renormalization is here done with a coefficient $1/d$.

Now, in the general case. We can remark that $[U^*]_{ij} = [U]_{ji}^*$. We also have:

$$\begin{aligned} \mathbf{d}U_{\mu\nu} &= i \sum_{\tau=1}^d \mathbf{d}H_{\mu\tau} U_{\tau\nu} - \frac{1}{2} U_{\mu\nu} \mathbf{d}t \\ \mathbf{d}\bar{U}_{\mu\nu} &= -i \sum_{\tau=1}^d \bar{U}_{\tau\nu} \mathbf{d}H_{\tau\mu} - \frac{1}{2} \bar{U}_{\mu\nu} \mathbf{d}t \end{aligned}$$

In turn this yields to the more general Lemma:

Lemma 2. *We have, for $\rho_1, \dots, \rho_r \in \mathcal{I}$:*

$$\begin{aligned} \mathbf{d}(U_{\rho_1} \dots U_{\rho_r}) &= -\frac{r}{2} U_{\rho_1} \dots U_{\rho_r} \mathbf{d}t \\ &+ \text{martingale part} - \frac{\mathbf{d}t}{nd} \sum_{1 \leq p < q \leq r} (-1)^{\omega_p + \omega_q} \zeta_{pq}^{(d)} \end{aligned}$$

where:

$$\zeta_{pq}^{(d)} = \begin{cases} U_{\rho_1} \dots U_{\mu_p \nu_q} \dots U_{\mu_q \nu_p} \dots U_{\rho_r} & \text{if } \omega_p = \omega_q = 0 \\ U_{\rho_1} \dots \bar{U}_{\mu_p \nu_q} \dots \bar{U}_{\mu_q \nu_p} \dots U_{\rho_r} & \text{if } \omega_p = \omega_q = 1 \\ \sum_{\tau=1}^{nd} \delta_{\mu_p \nu_q} U_{\rho_1} \dots \bar{U}_{\tau \nu_p} \dots U_{\tau \nu_q} \dots U_{\rho_r} & \text{if } \omega_p = 1, \omega_q = 0 \\ \sum_{\tau=1}^{nd} \delta_{\mu_p \mu_q} U_{\rho_1} \dots U_{\tau \nu_p} \dots \bar{U}_{\tau \nu_q} \dots U_{\rho_r} & \text{if } \omega_p = 0, \omega_q = 1 \end{cases} \quad (1)$$

Proof. It is an application of Itô's Lemma along with the observation that:

$$\mathbf{d}[U_{\mu\nu}, U_{\theta\eta}] = -\frac{\mathbf{d}t}{nd} U_{\theta\nu} U_{\mu\eta} \text{ and } \mathbf{d}[\bar{U}_{\mu\nu}, U_{\theta\eta}] = \sum_{\tau=1}^{nd} \frac{\mathbf{d}t}{nd} B_{\tau\nu} B_{\tau\eta} \delta_{\mu\theta}$$

□

So, taking up the same calculations as before, we get following system of differential equations:

$$\begin{aligned} \gamma'_{\alpha_{11}, \dots} &= -\frac{k_1 + \dots + k_r}{2} \gamma_{\alpha_{11}, \dots} \\ &- \sum_{\kappa=1}^r \sum_{1 \leq p < q \leq k_\kappa} (-1)^{\epsilon_{p\kappa} + \epsilon_{q\kappa}} \gamma_{(p, q, \kappa)} \\ &+ \mathcal{O}\left(\frac{1}{d^2}\right) \end{aligned}$$

where we note:

If $\epsilon_{p\kappa} = \epsilon_{q\kappa} = 0$:

$$\gamma_{(p, q, \kappa)} = \gamma_{\dots; \alpha_{1\kappa}, \dots, (i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa}), \alpha_{q+1, \kappa}, \dots; \alpha_{p+1, \kappa}, \dots, (i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa}); \dots}$$

That is, we have a switch exactly as before.

If $\epsilon_{p\kappa} = \epsilon_{q\kappa} = 1$:

$$\gamma_{(p,q,\kappa)} = \gamma_{\dots; \alpha_{1\kappa}, \dots, \alpha_{p-1,\kappa}, (i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa}), \dots; (i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa}), \dots; \dots}$$

That is, we also have here a switch as we have already seen.

If $\epsilon_{p\kappa} = 1, \epsilon_{q\kappa} = 0$:

$$\gamma_{(p,q,\kappa)} = \sum_{t=1}^n \delta_{i_{p\kappa} i_{q\kappa}} \gamma_{\dots; \alpha_{1\kappa}, \dots, (t j_{p\kappa} \epsilon_{p\kappa}), (t j_{q\kappa} \epsilon_{q\kappa}), \dots; \alpha_{p+1,\kappa} \dots \alpha_{q-1,\kappa}; \dots}$$

The structure is here a little more complicated, with a sum over t and t replacing the indices i_p and i_q and everything situated between the places p and q gets located in a new trace.

If $\epsilon_{p\kappa} = 0, \epsilon_{q\kappa} = 1$:

$$\gamma_{(p,q,\kappa)} = \sum_{t=1}^n \delta_{i_{p\kappa} i_{q\kappa}} \gamma_{\dots; \alpha_{1\kappa}, \dots, \alpha_{p-1,\kappa}, \alpha_{q+1,\kappa}, \dots; (t j_{p\kappa} \epsilon_{p\kappa}), \dots, (t j_{q\kappa} \epsilon_{q\kappa}); \dots}$$

the structure is almost the same as in the previous case, with the only difference that the places p and q and everything in between gets into a new trace.

3.3 A system of differential equations for the free stochastic process

We will now introduce:

$$\Gamma_{\alpha_{11}, \dots; \dots; \alpha_{1r}, \dots, \alpha_{kr}, r} = \phi(\Psi_{\alpha_{11}} \dots) \dots \phi(\Psi_{\alpha_{1r}} \dots \Psi_{\alpha_{kr}, r})$$

To prove the convergence of the marginals, we will show that Γ verifies the system of differential equations that we have just found, in the limit where d goes to infinity.

By using free stochastic calculus we can see that the quadratic variation is $\mathbf{d}X_{ij} \mathbf{d}X_{kl} = \delta_{il} \delta_{jk} \mathbf{d}t$. Moreover, the free stochastic differential equation yields, coefficient by coefficient:

$$\mathbf{d}\Psi_{uv} = \frac{i}{\sqrt{n}} \sum_{k=1}^n \mathbf{d}X_{uk} \Psi_{kv} - \frac{1}{2} \Psi_{uv} \mathbf{d}t$$

and

$$\mathbf{d}\Psi_{uv}^* = -\frac{i}{\sqrt{n}} \sum_{k=1}^n \Psi_{kv}^* \mathbf{d}X_{ku} - \frac{1}{2} \Psi_{uv}^* \mathbf{d}t$$

This allows us to prove following technical Lemma:

Lemma 3. For each $r \geq 2$ and all indices we have:

$$\begin{aligned} \mathbf{d}(\Psi_{\alpha_1} \dots \Psi_{\alpha_r}) &= -\frac{r \mathbf{d}t}{2} \Psi_{\alpha_1} \dots \Psi_{\alpha_r} \\ &+ \frac{i}{\sqrt{n}} \sum_{l=1}^r \sum_{k=1}^n (-1)^{\epsilon_l} \Psi_{\alpha_1} \dots \left\{ \begin{array}{ll} \mathbf{d}X_{i_l k} \Psi_{k j_l} & \text{if } \epsilon_l = 0 \\ \Psi_{k j_l}^* \mathbf{d}X_{k i_l} & \text{if } \epsilon_l = 1 \end{array} \right\} \dots \Psi_{\alpha_r} \\ &- \frac{\mathbf{d}t}{n} \sum_{1 \leq p < q \leq r} (-1)^{\epsilon_p + \epsilon_q} \zeta_{pq} \end{aligned}$$

where

$$\zeta_{pq} = \begin{cases} \Psi_{\alpha_1} \dots \Psi_{\alpha_{p-1}} \phi(\Psi_{i_q j_p}^{\epsilon_p} \dots \Psi_{\alpha_{q-1}}) \Psi_{i_p j_q}^{\epsilon_q} \dots & \text{if } \epsilon_p = \epsilon_q = 0 \\ \Psi_{\alpha_1} \dots \Psi_{\alpha_{p-1}} \Psi_{i_q j_p}^{\epsilon_p} \phi(\Psi_{\alpha_{p+1}} \dots \Psi_{\alpha_{q-1}} \Psi_{i_p j_q}^{\epsilon_q}) \Psi_{\alpha_{q+1}} \dots & \text{if } \epsilon_p = \epsilon_q = 1 \\ \sum_{k=1}^n \delta_{i_p i_q} \Psi_{\alpha_1} \dots \Psi_{\alpha_{p-1}} \phi(\Psi_{k j_p}^{\epsilon_p} \dots \Psi_{\alpha_{q-1}} \Psi_{k j_q}^{\epsilon_q}) \dots & \text{if } \epsilon_p = 0, \epsilon_q = 1 \\ \sum_{k=1}^n \delta_{i_p i_q} \Psi_{\alpha_1} \dots \Psi_{k j_p}^{\epsilon_p} \phi(\Psi_{\alpha_{p+1}} \dots \Psi_{\alpha_{q-1}}) \Psi_{k j_q}^{\epsilon_q} \dots & \text{if } \epsilon_p = 1, \epsilon_q = 0 \end{cases}$$

Proof. The proof is done by recurrence and by using Itô's formula. For simplicity's sake we will do it only in the case where all ϵ are put equal to zero.

For $r = 2$ we get:

$$\mathbf{d}(\Psi_{ij} \Psi_{kl}) = \frac{i}{\sqrt{n}} \sum_{s=1}^n \Psi_{ij} \mathbf{d}X_{ks} \Psi_{sl} + \frac{i}{\sqrt{n}} \sum_{s=1}^n \mathbf{d}X_{is} \Psi_{sj} \Psi_{kl} - \Psi_{ij} \Psi_{kl} \mathbf{d}t - \frac{\mathbf{d}t}{n} \phi(\Psi_{kj}) \psi_{il}$$

Hence we have the desired result for $r = 2$. Let us now assume that the Lemma is right until a certain r . Then, by Itô's Lemma:

$$\begin{aligned} \mathbf{d}(\Psi_{u_1 v_1} \dots \Psi_{u_{r+1} v_{r+1}}) &= -\frac{r+1}{2} \psi_{u_1 v_1} \dots \Psi_{u_{r+1} v_{r+1}} \mathbf{d}t \\ &+ \frac{i}{\sqrt{n}} \sum_{k=1}^n \psi_{u_1 v_1} \dots \Psi_{u_r v_r} \mathbf{d}X_{u_{r+1} k} \Psi_{k v_{r+1}} \\ &+ \frac{i}{\sqrt{n}} \sum_{k=1}^n \sum_{l=1}^r \Psi_{u_1 v_1} \dots \mathbf{d}X_{u_l k} \Psi_{k v_l} \dots \Psi_{u_{r+1} v_{r+1}} \\ &- \frac{\mathbf{d}t}{n} \sum_{1 \leq p < q \leq r} \Psi_{u_1 v_1} \dots \phi(\Psi_{u_q v_p} \dots) \Psi_{u_p v_q} \dots \Psi_{u_{r+1} v_{r+1}} \\ &- \frac{\mathbf{d}t}{n} \sum_{l=1}^r \Psi_{u_1 v_1} \dots \Psi_{u_{l-1} v_{l-1}} \phi(\Psi_{u_{r+1} v_l} \dots \Psi_{u_r v_r}) \Psi_{u_l v_{r+1}} \end{aligned}$$

And so we see that the result is also right for $r + 1$. \square

We now introduce, as expected, the family of functions:

$$\Gamma_{\alpha_{11}, \dots, \alpha_{1r}; \alpha_{1r}} = \phi(\Psi_{\alpha_{11}} \dots) \dots \phi(\Psi_{\alpha_{1r}} \dots)$$

By applying Lemma 3 we get:

$$\begin{aligned}\Gamma'_{\alpha_{11}, \dots, \dots; \alpha_{1r}, \dots} &= -\frac{k_1 + \dots + k_r}{2} \Gamma_{\alpha_{11}, \dots, \dots; \alpha_{1r}, \dots} \\ &\quad - \frac{1}{n} \sum_{\kappa=1}^r \sum_{1 \leq p < q \leq k_\kappa} (-1)^{\epsilon_p + \epsilon_q} \Gamma_{(p, q, \kappa)}\end{aligned}$$

where we defined:

$$\Gamma_{(p, q, \kappa)} = \begin{cases} \Gamma_{\dots; \alpha_{1\kappa}, \dots, i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa}, \dots, \alpha_{k_\kappa \kappa}; i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa}, \dots, \alpha_{q-1, \kappa}; \dots} & \text{if } \epsilon_{p\kappa} = \epsilon_{q\kappa} = 0 \\ \Gamma_{\dots; \alpha_{1\kappa}, \dots, i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa}, \alpha_{q+1, \kappa}, \dots, \alpha_{p+1, \kappa}, \dots, i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa}; \dots} & \text{if } \epsilon_{p\kappa} = \epsilon_{q\kappa} = 1 \\ \sum_{l=1}^n \delta_{i_{p\kappa} i_{q\kappa}} \Gamma_{\dots; \alpha_{1\kappa}, \dots, \alpha_{p-1, \kappa}, \alpha_{q+1, \kappa}, \dots, l j_{p\kappa} \epsilon_{p\kappa}, \dots, l j_{q\kappa} \epsilon_{q\kappa}; \dots} & \text{if } \epsilon_{p\kappa} = 0, \epsilon_{q\kappa} = 1 \\ \sum_{l=1}^n \delta_{i_{p\kappa} i_{q\kappa}} \Gamma_{\dots; \alpha_{1\kappa}, \dots, l j_{p\kappa} \epsilon_{p\kappa}, l j_{q\kappa} \epsilon_{q\kappa}, \dots, \alpha_{p+1, \kappa}, \dots, \alpha_{q-1, \kappa}; \dots} & \text{if } \epsilon_{p\kappa} = 1, \epsilon_{q\kappa} = 0 \end{cases}$$

Hence we see that the family of functions γ truly converges towards the family of functions Γ . In particular, taking $r = 1$, we see that the $*$ -moments of the family $(U_{ij}^{(d)})_{1 \leq i, j \leq n}$ converges towards the $*$ -moments of $(\Psi_{ij})_{1 \leq i, j \leq n}$. This proves the convergence of the marginals.

4 Conditional expectation

In order to prove Theorem 1 we must prove the convergence of all mixed moments of the kind: $\mathbb{E} \text{otr}(U_{i_1 j_1}^{\epsilon_1}(t_1) \dots U_{i_r j_r}^{\epsilon_r}(t_r))$ towards $\phi(\Psi_{i_1 j_1}^{\epsilon_1}(t_1) \dots \Psi_{i_r j_r}^{\epsilon_r}(t_r))$. In the previous section we have already proven that this is indeed the case when $\sharp\{t_1, \dots, t_r\} = 1$. In order to prove the general case we will use a method consisting of computing the joint moments by taking recursively conditional expectations.

4.1 Notations

In order to use this method, we must generalize somewhat our notations. In the sequel, we fix $s \geq 0$ and our time variable t will always verify $t \geq s$. We note:

1. by $[\mathcal{I}]$ the set $\{1, \dots, n\}^2 \times \{0, 1\} \times \mathcal{M}_d^{(s)}$, where $\mathcal{M}_d^{(s)}$ is the set of $d \times d$ matrices whose entries are \mathcal{F}_s -measurable random variables. Of course, we have $\mathcal{F}_s = \sigma(j_u, u \leq s)$.
2. by \mathcal{I} the set $\{1, \dots, nd\}^2 \times \{0, 1\} \times V^{(s)}$, where $V^{(s)}$ designates the set of \mathcal{F}_s -measurable random variables.
3. by \mathcal{I}^f the set $\{1, \dots, n\}^2 \times \{0, 1\} \times \mathcal{A}_s$, where \mathcal{A}_s is the $*$ -algebra generated by all $\Psi_{pq}(u), u \leq s$.

We use these sets as sets of indices in the following way:

1. If $\alpha = (i, j, \epsilon, m) \in [\mathcal{I}]$, we note $[U]_\alpha = m[U]_{ij}^\epsilon$
2. If $\rho = (\mu, \nu, \omega, \pi) \in \mathcal{I}$, we note $U_\rho = \pi U_{\mu\nu}^\omega$
3. If $\alpha = (i, j, \epsilon, m) \in \mathcal{I}^f$, we note $\Psi_\alpha = m\Psi_{ij}^\epsilon$.

4.2 A system of differential equations for the Brownian motion on $U(nd)$

We are interested in the family of functions:

$$\begin{aligned} & \gamma_{\alpha_{11}, \dots, \alpha_{k_1 1}; \dots; \alpha_{k_r r}}(t) \\ = & \mathbb{E}[tr([U]_{\alpha_{11}}(t) \dots [U]_{\alpha_{k_1 1}}(t)) \dots tr(\dots [U]_{\alpha_{k_r r}}(t))] \end{aligned}$$

In other words, we use the same family as before but we put \mathcal{F}_s -measurable elements between the blocks of the Brownian motion.

We want to use the same method as before. We will need following Lemma:

Lemma 4. *We have for any choice of indices in \mathcal{I} and for $t \geq s$:*

$$\begin{aligned} d(U_{\rho_1} \dots U_{\rho_k}) &= -\frac{k}{2} U_{\rho_1} \dots U_{\rho_k} dt \\ &\quad - \frac{1}{nd} \sum_{1 \leq p < q \leq k} (-1)^{\omega_p + \omega_q} \zeta_{pq}^{(d)} dt \\ &\quad + \text{martingale part} \end{aligned}$$

where:

$$\zeta_{pq}^{(d)} = \begin{cases} U_{\rho_1} \dots \pi_p U_{\mu_p \nu_q} \dots \pi_q U_{\mu_q \nu_p} \dots U_{\rho_k} & \text{if } \omega_p = \omega_q = 0 \\ U_{\rho_1} \dots \pi_p U_{\mu_p \nu_q}^* \dots \pi_q U_{\mu_q \nu_p}^* \dots U_{\rho_k} & \text{if } \omega_p = \omega_q = 1 \\ \sum_{\tau=1}^{nd} \delta_{\mu_p \mu_q} U_{\rho_1} \dots \pi_p U_{\tau \nu_p}^* \dots \pi_q U_{\tau \nu_q} \dots U_{\rho_k} & \text{if } \omega_p = 1, \omega_q = 0 \\ \sum_{\tau=1}^{nd} \delta_{\mu_p \mu_q} U_{\rho_1} \dots \pi_p U_{\tau \nu_p} \dots \pi_q U_{\tau \nu_q}^* \dots U_{\rho_k} & \text{if } \omega_p = 0, \omega_q = 1 \end{cases} \quad (2)$$

Proof. As always, this is proven using Itô's Lemma. \square

Applying this Lemma, we get:

Lemma 5. *The system of differential equations is:*

$$\begin{aligned} & \gamma'_{\alpha_{11}, \dots, \alpha_{k_1 1}; \dots; \alpha_{k_r r}} \\ = & -\frac{k_1 + \dots + k_r}{2} \gamma_{\alpha_{11}, \dots, \alpha_{k_1 1}; \dots; \alpha_{k_r r}} \\ & - \frac{1}{n} \sum_{\kappa=1}^r \sum_{1 \leq p < q \leq k_\kappa} (-1)^{\epsilon_{p\kappa} + \epsilon_{q\kappa}} \gamma_{(p, q, \kappa)} \\ & + \mathcal{O}\left(\frac{1}{d^2}\right) \end{aligned}$$

where:

$$\text{If } \epsilon_{p\kappa} = \epsilon_{q\kappa} = 0:$$

$$\gamma(p, q, \kappa) = \gamma_{\dots, \dots, (m_{p\kappa}, i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa}), \alpha_{q+1, \kappa} \dots; \alpha_{p+1, \kappa}, \dots, (m_{q\kappa}, i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa}); \dots}$$

$$\text{If } \epsilon_{p\kappa} = \epsilon_{q\kappa} = 1:$$

$$\gamma(p, q, \kappa) = \gamma_{\dots, \dots, (m_{p\kappa}, i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa}), \alpha_{q+1, \kappa}, \dots; \alpha_{p+1, \kappa}, \dots, (1, i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa}); \dots}$$

$$\text{If } \epsilon_{p\kappa} = 1, \epsilon_{q\kappa} = 0:$$

$$\gamma(p, q, \kappa) = \sum_{t=1}^n \delta_{i_{p\kappa} i_{q\kappa}} \gamma_{\dots, \dots, (m_{p\kappa}, t, j_{p\kappa} \epsilon_{p\kappa}), (t, j_{q\kappa}, \epsilon_{q\kappa}, 1), \dots; (i_{p+1, \kappa}, j_{p+1, \kappa}, \epsilon_{p+1, \kappa}, m_{q\kappa} m_{p+1, \kappa}), \dots; \dots}$$

$$\text{If } \epsilon_{p\kappa} = 0, \epsilon_{q\kappa} = 1:$$

$$\gamma(p, q, \kappa) = \sum_{t=1}^n \delta_{i_{p\kappa} i_{q\kappa}} \gamma_{\dots, \dots, (i_{q+1, \kappa}, j_{q+1, \kappa}, \epsilon_{q+1, \kappa}, m_{p\kappa} m_{q+1, \kappa}), \dots; (t, j_{p\kappa}, \epsilon_{p\kappa}, 1), \dots, (t, j_{q\kappa}, \epsilon_{q\kappa}, m_{q\kappa}); \dots}$$

The structure is very similar to what we had proved in the previous section. We just have to be careful to what happens with the m 's.

When we proved Biane's result we saw that the system of differential equations had a combinatorial structure related to the idea of integer partitions. I do not see any obvious combinatorial structure in this generalized formula but it is a question that is worth being asked.

4.3 A system of differential equations for the free stochastic process

Of course, we will be interested in the behavior of the family of functions:

$$\Gamma_{\alpha_{11}, \dots, \alpha_{k_1 1}; \dots} = \phi(\Psi_{\alpha_{11}}(t) \dots) \dots \phi(\dots)$$

Lemma 6. *For any choice of indices in \mathcal{I}^f and for $t \geq s$, we have:*

$$\begin{aligned} d(\Psi_{\alpha_1} \dots \Psi_{\alpha_k}) &= -\frac{k}{2} \Psi_{\alpha_1} \dots \Psi_{\alpha_k} dt \\ &+ \frac{i}{\sqrt{n}} \sum_{r=1}^n \sum_{l=1}^k \Psi_{\alpha_1} \dots \alpha_l \left\{ \begin{array}{ll} dX_{i_l r} \Psi_{r j_l} & \text{if } \epsilon_l = 0 \\ \Psi_{r j_l} dX_{r i_l} & \text{if } \epsilon_l = 1 \end{array} \right\} \dots \Psi_{\alpha_k} \\ &- \frac{dt}{n} \sum_{1 \leq p < q \leq k} (-1)^{\epsilon_p + \epsilon_q} \zeta_{pq} \end{aligned}$$

where

$$\zeta_{pq} = \begin{cases} \Psi_{\alpha_1} \dots \phi(\Psi_{i_q j_p}^{\epsilon_p} \dots \Psi_{\alpha_{q-1}} m_q) \Psi_{i_p j_q}^{\epsilon_q} \dots & \text{if } \epsilon_p = \epsilon_q = 1 \\ \Psi_{\alpha_1} \dots \alpha_p \Psi_{i_q j_p}^{\epsilon_p} \phi(\Psi_{\alpha_{p+1}} \dots \Psi_{i_p j_q}^{\epsilon_q}) \Psi_{\alpha_{q+1}} \dots & \text{if } \epsilon_p = \epsilon_q = 1 \\ \sum_{t=1}^k \delta_{i_p i_q} \Psi_{\alpha_1} \dots \alpha_p \phi(\Psi_{t j_p}^{\epsilon_p} \dots \Psi_{t j_q}^{\epsilon_q}) \Psi_{\alpha_{q+1}} \dots & \text{if } \epsilon_p = 0, \epsilon_q = 1 \\ \sum_{t=1}^k \delta_{i_p i_q} \Psi_{\alpha_1} \dots \Psi_{t j_p}^{\epsilon_p} \phi(\Psi_{\alpha_{p+1}} \dots \alpha_q) \Psi_{t j_q}^{\epsilon_q} \dots & \text{if } \epsilon_p = 1, \epsilon_q = 0 \end{cases}$$

Proof. It is the same proof as before, based on Itô's formula. \square

Applying this Lemma, we get:

Lemma 7. *The system of differential equations for the free stochastic process is:*

$$\begin{aligned} & \Gamma'_{\alpha_{11}, \dots, \dots} \\ &= -\frac{k_1 + \dots + k_r}{2} \Gamma_{\alpha_{11}, \dots, \dots} \\ &- \sum_{\kappa=1}^r \sum_{1 \leq p < q \leq k_\kappa} (-1)^{\epsilon_{p\kappa} + \epsilon_{q\kappa}} \Gamma_{(p, q, \kappa)} \end{aligned}$$

where:

$$\text{If } \epsilon_{p\kappa} = \epsilon_{q\kappa} = 0:$$

$$\Gamma_{(p, q, \kappa)} = \Gamma_{\dots, \dots, (i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa} m_{p\kappa}), \dots, (i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa} m_{q\kappa}), \dots, \alpha_{q-1, \kappa}, \dots}$$

$$\text{If } (\epsilon_{p\kappa}, \epsilon_{q\kappa}) = (1, 1):$$

$$\Gamma_{(p, q, \kappa)} = \Gamma_{\dots, \dots, (i_{q\kappa} j_{p\kappa} \epsilon_{p\kappa} m_{p\kappa}), \alpha_{q+1, \kappa}, \dots, \alpha_{p+1, \kappa}, \dots, (i_{p\kappa} j_{q\kappa} \epsilon_{q\kappa} 1), \dots}$$

$$\text{If } \epsilon_{p\kappa} = 0, \epsilon_{q\kappa} = 1:$$

$$\Gamma_{(p, q, \kappa)} = \sum_{t=1}^n \delta_{i_{p\kappa} i_{q\kappa}} \Gamma_{\dots, \dots, (i_{q+1, \kappa} j_{q+1, \kappa}, \epsilon_{q+1, \kappa}, m_{p\kappa} m_{q+1, \kappa}), \dots, (t j_{p\kappa} \epsilon_{p\kappa} 1), \dots, (t j_{q\kappa} \epsilon_{q\kappa} m_{q\kappa}), \dots}$$

$$\text{If } \epsilon_{p\kappa} = 1, \epsilon_{q\kappa} = 0:$$

$$\Gamma_{(p, q, \kappa)} = \sum_{t=1}^n \delta_{i_{p\kappa} i_{q\kappa}} \Gamma_{\dots, \dots, (t j_{p\kappa} \epsilon_{p\kappa} m_{p\kappa}), (t j_{q\kappa} \epsilon_{q\kappa} 1), \dots, (i_{p+1, \kappa} j_{p+1, \kappa}, \epsilon_{p+1, \kappa}, m_{q\kappa} m_{p+1, \kappa}), \dots, \dots}$$

4.4 Recurrence

We are now able to finish the proof of Theorem 1. We want to show that the moments $\mathbb{E} \text{otr}([U]_{i_1 j_1}(t_1)^{\epsilon_1} \dots [U]_{i_k j_k}(t_k)^{\epsilon_k})$ converge towards $\phi(\Psi_{i_1 j_1}^{\epsilon_1}(t_1) \dots \Psi_{i_k j_k}^{\epsilon_k}(t_k))$. Let us denote $\sigma = \sharp \{t_1, \dots, t_k\}$ the number of different times showing up in our moment. We are going to prove that result through recurrence on σ .

1. If $\sigma = 1$ the result has already been shown because it is just the convergence of the marginals.
2. Let us suppose that the result is true until a certain σ . We will now consider a moment using $\sigma + 1$ different times. We can order those times in increasing order: $t_1 \leq t_1 \leq \dots \leq t_{\sigma+1}$. The recurrence hypothesis tells us that:

$$(U_{p, q}(t_i))_{\substack{1 \leq i \leq \sigma \\ 1 \leq p, q \leq n}} \xrightarrow{\text{in } * \text{-moments}} (\Psi_{p, q}(t_i))_{\substack{1 \leq i \leq \sigma \\ 1 \leq p, q \leq n}}$$

We can write the moment under consideration as:

$$\gamma_{(i_1 j_1 \epsilon_1 m_1^{(d)}), \dots, (i_k j_k \epsilon_k m_k^{(d)})}(t_{\sigma+1})$$

where the $m_i^{(d)}$ are \mathcal{F}_{t_σ} -measurable. Now, let us remark that the family of functions $(\gamma_{\alpha_{11}, \dots, \alpha_{k_1 1}; \alpha_{12}, \dots})$ is entirely characterized by the system of differential equations from Lemma 5 along with all the relationships between the $\{m_{ij}^{(d)}, 1 \leq j \leq r, 1 \leq i \leq k_j\}$. In the same way, the family Γ_{\dots} is entirely defined by the system from Lemma 7 along with the relationships between the $\{m_{ij}, 1 \leq j \leq r, 1 \leq i \leq k_j\}$

Now, the recurrence hypothesis allows us to say that the $m_i^{(d)}, 1 \leq i \leq k$ converges towards some $m_i, 1 \leq i \leq k$. This tells us that the relationships between the $\{m_i^{(d)}\}$ "converges" towards the relationships between the $\{m_i\}$. Moreover, the system of differential equations from Lemma 5 converges towards that of Lemma 7. To put it in a nutshell, this means:

$$\gamma_{\alpha_1^{(d)}, \dots, \alpha_1^{(d)}}(t_{\sigma+1}) \xrightarrow{d \rightarrow \infty} \Gamma_{\alpha_1, \dots, \alpha_k}(t_{\sigma+1})$$

Or, in other words, we have the convergence of our moment.

Thus, we have proven that all *-moments converge and this means that Theorem 1 is proven.

5 Some examples of calculations and gaussianity

We will now use the differential equations that we obtained to calculate some simple moments of our process. We will then be able to draw a consequence about the gaussianity of the free process. In the sequel, we denote by ϕ_t the function defined on $U\langle n \rangle$ by $\phi_t = \phi \circ J_{0t}$ where J_t is the limit (free) process.

5.1 The first moments

Let us take now $1 \leq i \neq j \leq n$. We have the following differential equations:

$$\begin{aligned} \frac{d}{dt} \phi_t(u_{ii}) &= -\frac{1}{2} \phi_t(u_{ii}) \\ \frac{d}{dt} \phi_t(u_{ij}) &= -\frac{1}{2} \phi_t(u_{ij}) \end{aligned}$$

with initial conditions: $\phi_0(u_{ii}) = 1$ and $\phi_0(u_{ij}) = 0$. It thus yields:

$$\begin{aligned} \phi_t(u_{ii}) &= e^{-\frac{1}{2}t} \\ \phi_t(u_{ij}) &= 0 \end{aligned}$$

We find the same expression for $\phi_t(u_{ii}^*)$ and $\phi_t(u_{ij}^*)$ because they obey the same differential equation with the same initial conditions.

5.2 The second moments

Let us take $1 \leq i, j, k, l \leq n$. We have following equation:

$$\begin{aligned} \frac{d}{dt} \phi_t(u_{ij}u_{kl}) &= -\phi_t(u_{ij}u_{kl}) - \phi_t(u_{il})\phi_t(u_{kj})\frac{1}{n} \\ &= -\phi_t(u_{ij}u_{kl}) - \frac{1}{n}\delta_{il}\delta_{kj}e^{-t} \end{aligned}$$

with initial conditions $\phi_0(u_{ij}u_{kl}) = \delta_{ij}\delta_{kl}$ because $\Psi_0 = I$. This equation is a linear differential equation of order 1 and the well-known method allows us to say:

$$\phi_t(u_{ij}u_{kl}) = \frac{\delta_{ij}\delta_{kl}}{n}e^{-t} - t\delta_{il}\delta_{kj}e^{-t}$$

The moments $\phi_t(u_{ij}^*u_{kl}^*)$ also obey the same equation with the same initial condition and they therefore have the same expression. If we are interested in $\phi_t(u_{ij}u_{kl}^*)$ we get the equation:

$$\frac{d}{dt} \phi_t(u_{ij}u_{kl}^*) = -\phi_t(u_{ij}u_{kl}^*) + \frac{1}{n} \sum_{p=1}^n \phi_t(u_{pj}u_{pl}^*)$$

with initial conditions $\phi_0(u_{ij}u_{kl}^*) = \delta_{ij}\delta_{kl}$. This can be put in the form of a system of linear differential equations by putting $\Phi_t = (\phi_t(u_{ij}u_{kl}))_{1 \leq i,j,k,l \leq n}$ seen as a vector of \mathbb{C}^{n^4} and $A = (a_{(r_1,r_2,r_3,r_4),(s_1,s_2,s_3,s_4)})$ as a matrix acting on \mathbb{C}^{n^4} , with:

$$\begin{cases} a_{rs} = 0 & \text{if } s_1 = s_3 \text{ and } r = s \\ a_{rs} = 1/n & \text{if } s_1 = s_3 \text{ and } r \neq s \\ a_{rs} = -1 & \text{if } r = s \text{ and } r_1 \neq r_3 \end{cases}$$

The equation then is:

$$\Phi' = A\Phi$$

The solution of such an equation is of the form $\Phi_t = Ce^{At}$ with C a constant.

5.3 Gaussianity

We would like to define a Brownian motion on $U\langle n \rangle$ as a free stochastic process having the same law (the same *-moments) as Ψ_t . This would seem natural because it is just the limit of the Brownian motion on $U(nd)$. To know if this definition makes sense, we would like Ψ_t to verify some properties, and especially the gaussian property as defined in [2], Proposition 1.12 and in [7], Proposition 5.1.1.

We define a counit δ on $U\langle n \rangle$ as the morphism of *-algebras verifying $\delta(u_{ij}) = \delta_{ij}$. We recall following definition and results from [2] and from [7]:

Definition 6 (Definition 1.8 from [2]). Let \mathcal{B} be a unital $*$ -algebra equipped with a character $\delta : \mathcal{B} \rightarrow \mathbb{C}$. A Schürmann triple on (\mathcal{B}, ϵ) is a triple (π, η, L) consisting of:

- A unital $*$ -representation $\pi : \mathcal{B} \rightarrow L(D)$ on some pre-Hilbert space D .
- A linear map $\eta : \mathcal{B} \rightarrow D$ verifying:

$$\eta(ab) = \pi(a)\eta(b) + \eta(a)\epsilon(b)$$

- A hermitian linear functional $L : \mathcal{B} \rightarrow \mathbb{C}$ such that:

$$-\langle \eta(a^*), \eta(b) \rangle = \epsilon(a)L(b) - L(ab) + L(a)\epsilon(b)$$

Property 1 (Theorem 1.9 from [2]). There is a one-to-one correspondence between Schürmann triples, generators of Lévy processes and Lévy processes.

Definition 7 (Proposition 5.1.1 from [7]). We say that a Lévy process on $U\langle n \rangle$ is gaussian if one of the following equivalent properties are verified:

- For each $a, b, c \in \text{Ker}\delta$, we have $L(abc) = 0$.
- For each $a, b \in \text{Ker}\delta$, we have $L(b^*a^*ab) = 0$.
- For all $a, b, c \in U\langle n \rangle$ we have the following formula:

$$\begin{aligned} L(abc) &= L(ab)\delta(c) + L(ac)\delta(b) + \delta(a)L(bc) - \delta(a)\delta(b)L(c) \\ &\quad - \delta(a)\delta(c)L(b) - L(a)\delta(b)\delta(c) \end{aligned}$$

- The representation π is zero on $\text{Ker}\delta$: $\pi|_{\text{Ker}\delta} = 0$.
- We have for each $a \in U\langle n \rangle$: $\pi(a) = \delta(a)Id$.
- For each a, b in $\text{Ker}\delta$, we have: $\eta(ab) = 0$.
- We have for all a, b in $U\langle n \rangle$: $\eta(ab) = \delta(a)\eta(b) + \eta(a)\delta(b)$.

Theorem 2. Let us take $D = \mathcal{M}_n(\mathbb{C})$. We then define a Schürmann triple by setting:

$$\eta(u_{jk}) = \epsilon_{jk}/\sqrt{n}, \eta(u_{jk}^*) = -\epsilon_{kj}\sqrt{n}$$

$$\pi(u_{jk}) = \delta_{jk}Id$$

$$L(u_{jk}) = -\frac{1}{2} \sum_{r=1}^n \langle \eta(u_{rj}^*), \eta(u_{rk}) \rangle$$

where ϵ_{jk} describe the elementary matrices.

Then, the Schürmann triple (η, π, L) is associated to the Lévy process on $U\langle n \rangle$ we are interested in.

Proof. We prove it by recurrence on the length of the words:

For the length 1: we have:

$$L(u_{jk}) = -\frac{1}{2} \sum_{r=1}^n \langle \epsilon_{rj}, \epsilon_{rk} \rangle = -\frac{1}{2n} \sum_{r=1}^n \text{Tr}(\epsilon_{jr} \epsilon_{rk}) = -\delta_{jk}/2$$

Let us suppose the result is true for words of length up to k : We must first find an expression for η . The cocycle property for η allows us to find through an easy recurrence that:

$$\eta(u_{i_1 j_1}^{\epsilon_1} \dots u_{i_k j_k}^{\epsilon_k}) = \sum_{p=1}^k \delta_{i_1 j_1} \dots \begin{cases} \epsilon_{i_p j_p} & \text{if } \epsilon_p = 0 \\ -\epsilon_{j_p i_p} & \text{if } \epsilon_p = 1 \end{cases} \dots \delta_{i_k j_k}$$

We can now use the coboundary property to write:

$$\begin{aligned} L(u_{i_1 j_1}^{\epsilon_1} \dots u_{i_{k+1} j_{k+1}}^{\epsilon_{k+1}}) &= \epsilon(u_{i_2 j_2}^{\epsilon_2} \dots u_{i_{k+1} j_{k+1}}^{\epsilon_{k+1}}) L(u_{i_1 j_1}^{\epsilon_1}) + L(u_{i_2 j_2}^{\epsilon_2} \dots u_{i_{k+1} j_{k+1}}^{\epsilon_{k+1}}) \epsilon(u_{i_1 j_1}^{\epsilon_1}) \\ &+ \langle \eta(u_{i_1 j_1}^{1-\epsilon_1}), \eta(u_{i_2 j_2}^{\epsilon_2} \dots u_{i_k j_k}^{\epsilon_k} u_{i_{k+1} j_{k+1}}^{\epsilon_{k+1}}) \rangle \\ &= -\frac{k}{2} \delta_{i_1 j_1} \dots \delta_{i_k j_k} \delta_{i_{k+1} j_{k+1}} \\ &- \sum_{2 \leq p < q \leq k+1} (-1)^{\epsilon_p + \epsilon_q} \Gamma_{(p,q,1)} \delta_{i_1 j_1} \\ &- \delta_{i_1 j_1} \delta_{i_2 j_2} \dots \delta_{i_{k+1} j_{k+1}} / 2 \\ &+ \clubsuit \end{aligned}$$

where we have used the fact that the Brownian motion on $U(nd)$ at time $t = 0$ is just Id . So we only have to compute the value of \clubsuit , which is the term arising from $\langle \eta(u_{i_1 j_1}^{1-\epsilon_1}), \eta(u_{i_2 j_2}^{\epsilon_2} \dots u_{i_k j_k}^{\epsilon_k} u_{i_{k+1} j_{k+1}}^{\epsilon_{k+1}}) \rangle$. We also remark that to finish our recurrence, it suffices to show that this \clubsuit is equal to

$$- \sum_{2 \leq p \leq k+1} (-1)^{\epsilon_p + \epsilon_1} \Gamma_{(1,p,1)}$$

So we may now write:

$$\begin{aligned} &\langle \eta(u_{i_1 j_1}^{1-\epsilon_1}), \eta(u_{i_2 j_2}^{\epsilon_2} \dots u_{i_k j_k}^{\epsilon_k} u_{i_{k+1} j_{k+1}}^{\epsilon_{k+1}}) \rangle \\ &= \frac{1}{n} \left\langle \begin{cases} -\epsilon_{j_1 i_1} & \text{if } \epsilon_1 = 0 \\ \epsilon_{i_1 j_1} & \text{if } \epsilon_1 = 1 \end{cases}, \sum_{p=2}^{k+1} \delta_{i_2 j_2} \dots \begin{cases} \epsilon_{i_p j_p} & \text{if } \epsilon_p = 0 \\ -\epsilon_{j_p i_p} & \text{if } \epsilon_p = 1 \end{cases} \dots \delta_{i_{k+1} j_{k+1}} \right\rangle \\ &= \sum_{p=2}^{k+1} \spadesuit_p \end{aligned}$$

We may now study the four cases:

Case where $\epsilon_1 = \epsilon_p = 0$: we have

$$\spadesuit_p = -\frac{1}{n} \delta_{i_1 j_p} \delta_{i_2 j_2} \dots \delta_{i_p j_1} \dots = -(-1)^{\epsilon_1 + \epsilon_p} \Gamma_{(1,p,1)}$$

Case where $\epsilon_1 = \epsilon_p = 1$: we have

$$\spadesuit_p = -\frac{1}{n}\delta_{i_1 j_p}\delta_{i_2 j_2}\dots\delta_{i_p j_1}\dots = -(-1)^{\epsilon_1+\epsilon_p}\Gamma_{(1,p,1)}$$

Case where $\epsilon_1 = 0, \epsilon_p = 1$: we have

$$\spadesuit_p = \frac{1}{n}\delta_{i_1 i_p}\delta_{i_2 j_2}\dots\delta_{j_p j_1}\dots = -(-1)^{\epsilon_1+\epsilon_p}\Gamma_{(1,p,1)}$$

Case where $\epsilon_1 = 1, \epsilon_p = 0$: we have

$$\spadesuit_p = \frac{1}{n}\delta_{i_1 i_p}\delta_{i_2 j_2}\dots\delta_{j_p j_1}\dots = -(-1)^{\epsilon_1+\epsilon_p}\Gamma_{(1,p,1)}$$

Thus, we have proven the result by recurrence. \square

Theorem 3. *The Lévy process from Theorem 1 is gaussian.*

Proof. It is immediate by using the fifth characterization from Definition 6. \square

Our Lévy process is thus a good candidate to define what we would like to call a Brownian motion on $U\langle n \rangle$.

6 Conclusion

We have proven in this article a generalization of Biane's result, namely that the Brownian motion on $U(nd)$, seen block-wise, converges towards a Lévy process on the Unitary Dual Group $U\langle n \rangle$, as d goes to infinity. Biane's result can thus be seen as a Lévy process on $U\langle 1 \rangle$. The proof of our generalized result uses quite elementary tools, ie mainly the convergence of systems of differential equations and combinatorial considerations.

This limit free Lévy process is described by using a free stochastic differential equation whose form is similar to the equation of the Brownian motion on $U(nd)$. A natural question would be to know if other classical matricial Lévy processes arising from (classical) stochastic equations yield (free) Lévy process described by a similar (free) stochastic equation.

Also, this free Lévy process seems to be a good definition for a Brownian motion on our dual group $U\langle n \rangle$.

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